Restrictions on spontaneous symmetry breaking in gauge theories with massive fermions

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Vafa and Witten have shown that massive fermions can hinder a global continuous symmetry to be spontaneously broken.[1] The main purpose of this paper is to present their proof. However, we do not follow the logic of their paper. The proof depends on properties of the Goldstone boson and the Dirac operator. These properties are not only technicalities in the proof, but they are interesting in their own right. That is why we summarize them separately, and then complete the proof of the theorem. To make this presentation self-contained we start it with a review on the basics of spontaneous symmetry breaking.

I. ASPECTS OF SPONTANEOUS SYMMETRY BREAKING

A. Effective action and its symmetries

The vacuum-vacuum amplitude of a quantum field theory with action $I[\phi]$ in the presence of a set of classical currents J_r coupled to the fields ϕ^r is given by[2]

$$Z[J] := \langle \text{vac}, \text{out} | \text{vac}, \text{in} \rangle_J := \int \prod_{s,y} d\phi^s(y) e^{iI[\phi]+i\int d^4x\, \phi^r(x) J_r(x) + \epsilon \text{ terms}}, \tag{1}$$

where ϕ^r can live in any representation of the Lorentz group, and can be even fermionic. If ϕ^r is fermionic, then J_r is also fermionic in order to get a bosonic action, and in this case we have to keep track of the order of the fields in the functional derivatives. δ_L and δ_R will indicate that before performing the differentiation we arrange the field with respect to which we are differentiating to the left or to the right, respectively. The ϵ terms just have the effect of putting the correct $i\epsilon$ in the denominators of all propagators. From now on we omit these terms. In terms of the sum of all connected vacuum amplitudes, $iW[J]$ ($Z[J] = \exp(iW[J])$), the vacuum expectation value of the operator $\Phi^r(x)$ in the presence of the current J ($J = (J_r)_{r=1,\dots,N}$, N is the number of the various fields):

$$\phi^r_J(x) := \frac{\langle \text{vac}, \text{out} | \Phi^r(x) | \text{vac}, \text{in} \rangle_J}{\langle \text{vac}, \text{out} | \text{vac}, \text{in} \rangle_J} = \frac{\delta_R W[J]}{\delta J_r(x)}. \tag{2}$$

Now, define J^ϕ as the current for which ϕ^r_J has the prescribed value ϕ^r:

$$\phi^r_{J^\phi} = \phi^r. \tag{3}$$

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The quantum effective action is defined by

$$\Gamma[\phi] := W[J^\phi] - \int d^4x\, \phi^r(x) J_r^\phi(x). \tag{4}$$

Using (2)

$$\frac{\delta_L \Gamma[\phi]}{\delta \phi^r(x)} = -J_r^\phi(x).$$

Does a classical symmetry survive quantization? Does Γ inherit the symmetries of I? We are primarily interested in global linear symmetries ($(M\phi)_n(x) = M_{nm}\phi_m(x)$), so instead of deriving what symmetry conditions are generally imposed on Γ by the symmetries of I (known as Slavnov-Taylor identities), we can check quickly that these symmetries are preserved by Γ. (We assume that M does not mix bosonic and fermionic fields, so all the entries of M are ordinary numbers.) If $I[M\phi] = I[\phi]$, and

$$\prod_{s,y} d(M^{-1}\phi)^s(y) = \prod_{s,y} d\phi^s(y), \tag{5}$$

then by changing variables in (1) we can establish from (2) and (3) that

$$W[J] = W[M^{-1^T} J], \quad M\phi_J = \phi_{M^{-1^T} J}, \quad J^{M\phi} = M^{-1^T} J^\phi,$$

so from (4) we obtain

$$\Gamma[M\phi] = \Gamma[\phi]. \tag{6}$$

Actually, this argument should not be considered as a proof of the invariance of Γ. It would be a proof if we could always define the path integral in (1). What usually happens is that we require the invariance of Γ, and (for instance, using the Salvnov-Taylor identities as a guide), we define (1) so that the manipulations that led to the invariance of Γ are permissible. Sometimes this program cannot be carried out, for example when the theory is simply nonrenormalizable, or when any regularization of the path integral ruins the invariance of the measure ((5) does not hold).

B. Translation invariant vacua

If the field operator Φ ($\Phi = (\Phi_r)_{r=1,\dots,N}$) has constant expectation value $\bar\phi$ ($\bar\phi = (\bar\phi_r)_{r=1,\dots,N}$) over a large spacetime volume $V_3 T$, then it can be shown that

$$\min\left\{ \langle \Omega_{\bar\phi} | H | \Omega_{\bar\phi} \rangle \;\middle|\; \langle \Omega_{\bar\phi} | \Omega_{\bar\phi} \rangle,\ \langle \Omega_{\bar\phi} | \Phi | \Omega_{\bar\phi} \rangle = \bar\phi \right\} = -\frac{1}{T}\Gamma[\bar\phi], \tag{7}$$

where H is the Hamiltonian. If Ω is a state for which $\langle \Omega | H | \Omega \rangle$ is an (absolute) minimum (such an Ω is called vacuum), and $\langle \Omega | \Phi(x) | \Omega \rangle = \bar\phi$ is constant, and (6) holds for some M, then we can see from (7) that $\langle \Omega | \Phi | \Omega \rangle = \langle \Omega' | \Phi | \Omega' \rangle$ for a state

Ω' which gives $\langle\Omega'|\Phi|\Omega'\rangle = M\bar{\phi}$. If $M\bar{\phi} \neq \bar{\phi}$, then $\Omega' \neq \Omega$, so the vacuum is degenerate. Of course, we cannot conclude that the symmetry is broken. The true vacuum can be a linear combination of the symmetry breaking vacua.

These calculations were valid only with a finite spacetime volume regularization. If we want to prove that spontaneous symmetry breaking really happens, we have to leave behind this regularization. So let us suppose that we found that in the infinite spacetime volume limit we still have degenerate vacua Ω_v. Assume that these vacua are invariant under spatial translations: $\mathbf{P}|\Omega_v\rangle = 0$ ($\mathbf{P}$ is the three-momentum operator). Let us choose an orthonormal set of vacua:

$$\langle\Omega_u|\Omega_v\rangle = \delta_{uv}.$$

Any matrix element of the product of two operators at equal times between these states (using the translation invariance of the vacuum states):

$$\langle\Omega_u|A(\mathbf{x})B(\mathbf{0})|\Omega_v\rangle = \\ \sum_w \langle\Omega_u|A(\mathbf{0})|\Omega_w\rangle\langle\Omega_w|B(\mathbf{0})|\Omega_v\rangle + \int d^3\mathbf{p} \sum_N \langle\Omega_u|A(\mathbf{0})|N_p\rangle\langle N_p|B(\mathbf{0})|\Omega_v\rangle e^{-i\mathbf{p}\cdot\mathbf{x}}. \tag{8}$$

(We suppressed the time variable.) $|N_p\rangle$ are orthonormalized states of three-momentum $\mathbf{p}$, which together with the vacuum states span the whole physical Hilbert space. The point of (8) is that for the single and the multiparticle states $|N_p\rangle$ the three-momentum $\mathbf{p} = 0$ is the part of the continuous spectrum as opposed to the vacuum states for which $\mathbf{p} = 0$ is a discrete eigenvalue. (At this point it is necessary to assume that the space volume is infinite, otherwise $\mathbf{p} = 0$ would be a discrete eigenvalue even for the single/multiparticle sates, and the vacuum states would not be distinguished in this respect.) Hence it is plausible to suppose that the expectation values in the second sum on the right hand side of (8) are smooth enough functions of $\mathbf{p}$ so that we can apply (a variant of) the Riemann-Lebesgue theorem to claim that the integral over $\mathbf{p}$ vanishes as $|\mathbf{x}| \to \infty$. Thus we have

$$\lim_{|\mathbf{x}|\to\infty} \langle\Omega_u|A(\mathbf{x})B(\mathbf{0})|\Omega_v\rangle = \sum_w \langle\Omega_u|A(\mathbf{0})|\Omega_w\rangle\langle\Omega_w|B(\mathbf{0})|\Omega_v\rangle.$$

For local operators $[A(\mathbf{x}), B(\mathbf{0})] = 0$ if $\mathbf{x} \neq 0$, so repeating the same calculation for $\lim_{|\mathbf{x}|\to\infty}\langle\Omega_u|B(\mathbf{0})A(\mathbf{x})|\Omega_v\rangle$, we get that the matrices $\langle\Omega_u A(\mathbf{0})\Omega_v\rangle$ and $\langle\Omega_u B(\mathbf{0})\Omega_v\rangle$ commute with one another. It follows that there is a basis $\{\Omega_u^0\}$ in which every Hermitian local operator $A(\mathbf{x})$ of the theory is diagonal:

$$\langle\Omega_u^0|A(\mathbf{x})|\Omega_v^0\rangle = \delta_{uv}a_v.$$

A symmetry breaking pertubation B built out of local operators will be diagonal in the same basis as the Hamiltonian. A general symmetry breaking operator has no matrix element which is invariant under the symmetry transformation M, so none of the matrix elements in the basis $\{\Omega_u^0\}$ are invariant under M. In the presence of the perturbation B the system will prefer a symmetry breaking state as its vacuum state (if all the eigenvalues of the perturbed Hamiltonian are different, which is the general case).

C. Goldstone bosons

We restrict our attention to the spontaneous breaking of a continuous symmetry generated by t. Such a symmetry of the action leads to the existence of a conserved current J^μ:

$$\partial_\mu J^\mu(x) = 0. \tag{9}$$

The charge that induces the associated symmetry transformation:

$$Q := \int d^3\mathbf{x}\ J^0(\mathbf{x}, 0), \tag{10}$$

and

$$[Q, \phi_n(x)] = -\sum_m t_{nm}\phi_m(x). \tag{11}$$

Here ϕ_n is a scalar field. (We are interested in such symmetry breaking where the vacua are Poincaré invariant, so the symmetry breaking can be manifested only in a scalar field's noninvariant vacuum expectation value.) The operator relations (9) and (11) are preserved by the the spontaneous breaking, which affects only the physical *states*. The vacuum expectation value of the commutator of the current and the scalar fields (from the translational invariance of the vacuum):

$$\langle [J^\lambda(y), \phi_n(x)]\rangle_{\text{vac}} = \frac{-i}{(2\pi)^3}\int d^4p\ [\rho_n^\lambda(p)e^{ip\cdot(y-x)} - \tilde{\rho}_n^\lambda(p)e^{ip\cdot(x-y)}], \tag{12}$$

where

$$\rho_n^\lambda(p) = (2\pi)^3 i\sum_N \langle \text{vac}|J^\lambda(0)|N_{p_N}\rangle\langle N_{p_N}|\phi_n(0)|\text{vac}\rangle\delta^{(4)}(p - p_N), \tag{13}$$

and $\tilde{\rho}_n^\lambda$ is defined similarly with ϕ_n and J^λ exchanged. Translational invariance of the vacuum tells us

$$\rho_n^\lambda(p) = p^\lambda\rho_n(-p^2)\theta(p^0), \quad \tilde{\rho}_n^\lambda(p) = p^\lambda\tilde{\rho}_n(-p^2)\theta(p^0).$$

The factor $\theta(p^0)$ appears because the matrix elements are taken between physical states (which have positive energy). In terms of the scalar spectral function (ρ_n and $\tilde{\rho}_n$) (12) reads

$$\langle [J^\lambda(y), \phi_n(x)]\rangle_{\text{vac}} = \frac{\partial}{\partial y_\lambda}\int d\mu^2 \left[\rho_n(\mu^2)\Delta_+(y - x; \mu^2) + \tilde{\rho}_n(\mu^2)\Delta_+(x - y; \mu^2)\right], \tag{14}$$

where

$$\Delta_+(z; \mu^2) = \frac{1}{(2\pi)^3}\int d^4p\ \theta(p^0)\delta(p^2 + \mu^2)e^{ip\cdot z}.$$

For spacelike z we have $\Delta_+(z; \mu^2) = \Delta_+(-z; \mu^2)$, from which and (14) we get

$$\rho_n(\mu^2) = -\tilde{\rho}_n(\mu^2), \tag{15}$$

because the commutator must vanish for spacelike $x-y$. Differentiating (14) with respect ot y^λ, using (9) and $(\Box_y - \mu^2)\Delta_+(y-x;\mu^2) = 0$ (if $x \neq y$), we find that for *all*, even for timelike x and y ($x \neq y$)

$$\int d\mu^2 \mu^2 \rho_n(\mu^2)\left[\Delta_+(y-x;\mu^2) - \Delta_+(x-y,\mu^2)\right] = 0,$$

and so

$$\mu^2 \rho_n(\mu^2) = 0. \tag{16}$$

Setting $\lambda = 0$, $x^0 = y^0 = t$ in (12), using (15),

$$\begin{aligned}\langle [J^0(\mathbf{y},t), \phi_n(\mathbf{x},t)]\rangle_{\text{vac}} &= \frac{2i}{(2\pi)^3}\int d\mu^2 \rho_n(\mu^2)\int d^4p\,\sqrt{\mathbf{p}^2+\mu^2}e^{i\mathbf{p}\cdot(\mathbf{y}-\mathbf{x})}\delta(p^2+\mu^2)\\ &= i\delta^{(3)}(\mathbf{y}-\mathbf{x})\int d\mu^2\rho_n(\mu^2).\end{aligned}$$

Integrating over $\mathbf{y}$, using (10) and (11), we obtain

$$-\sum_m t_{nm}\langle\phi_m(x)\rangle_{\text{vac}} = i\int d\mu^2 \rho_n(\mu^2). \tag{17}$$

The solution to (16) and (17) is

$$\rho_n(\mu^2) = i\delta(\mu^2)\sum_m t_{nm}\langle\phi_m(x)\rangle_{\text{vac}}. \tag{18}$$

(Note that this derivaton is valid only for constant $\langle\phi_n(x)\rangle_{\text{vac}}$ because we used the translation invariance of the vacuum.) In (13) when N labels multiparticle states, it includes an integration over at least two three-momenta, so we do not get a Dirac delta. (18) indicates the existence of a massless single particle state, which has zero helicity because $\phi_n(0)|\text{vac}\rangle$ is rotationally invariant, so $\langle N|\phi_n(0)|\text{vac}\rangle$ vanishes for any state of nonzero helicity. The virtue of this derivation is not only that it predicts the existence of such a state when a continuous global symmetry is spontaneously broken, but it makes it clear that the current associated with the broken symmetry has nonvanishing matrix element between this state and the vacuum.

D. Two-point function of currents

What can be inferred from the behavior of the two point function of J^μ_a about the existence of a one particle state B for which $\langle\text{vac}|J^\mu_a|B\rangle \neq 0$? Considering that

$$\langle J^\mu_a(x)J^\nu_b(0)\rangle_{\text{vac}} = \sum_N \langle\text{vac}|J^\mu_a(x)|N\rangle\langle N|J^\nu_b(0)|\text{vac}\rangle, \tag{19}$$

it is useful to evaluate $\langle\text{vac}|J^\mu_a|B^s_p\rangle$, where B^s_p is a one particle state of momentum p and third spin component or helicity s. ($\langle B^s_p, B^{s'}_{p'}\rangle = \delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta^{ss'}$.) Poincaré invariance of the vacuum restricts the possible spin or helicity of B^s_p for which the above-mentioned matrix element does not vanish. Generally, let us consider an operator O_ℓ living in the representation S of the Lorentz group, i.e., for translations a and Lorentz transformations Λ:[3]

$$U(a)O_\ell U(a)^+ = O_\ell(x+a), \quad U(\Lambda)O_\ell(x)U(\lambda)^+ = \sum_{\bar{\ell}} S_{\ell\bar{\ell}}(\Lambda^{-1})O_{\bar{\ell}}(\Lambda x), \tag{20}$$

where U is the unitary (ray) representation of the Poincaré group on the physical Hilbert space under which B^s_p are transformed as

$$U(a)B^s_p = e^{-ia\cdot p}B^s_p, \quad U(\Lambda)B^s_p = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{\bar{s}} D^{\bar{s}s}(W(\Lambda,p))B^{\bar{s}}_{\Lambda p}. \tag{21}$$

$W(\Lambda, p)$ is the Wigner-rotation ($W(\Lambda, p) = L(\Lambda p)^{-1}\Lambda L(p)$, where $L(p)$ is a set of Lorentz transformation such that $L(p)k = p$ for some standard momentum k.) Using again the Poincaré invariance of the vacuum with (20) and (21), we have

$$\langle \mathrm{vac}|O_\ell|B^s_p\rangle = \frac{e^{ip\cdot x}}{\sqrt{p^0}} f^s_\ell(p), \tag{22}$$

where $f^s_\ell(p)$ satisfies

$$f(\Lambda p) = D^*(W(\Lambda, p)) \otimes S(\Lambda)\ f(p). \tag{23}$$

In the derivation of (23) we used the unitarity of D. Equation (23) provides a functional equation for f which can easily be solved. Indeed, setting $p = k$, $\Lambda = L(p)$,

$$f^s_\ell(p) = S(L(p))_{\ell\bar{\ell}}\ f^s_{\bar{\ell}}(p), \tag{24}$$

which defines $f(p)$ for every p. Of course, this definition is good (independent of the choice of $L(p)$) only if

$$D^*(\lambda) \otimes S(\lambda)\ f(k) = f(k) \quad \text{for any } \lambda \text{ from the little group of } k\ (\lambda k = k).$$

Now, let O_ℓ be a Lorentz vector (as J^μ_a), so $S(\Lambda) = \Lambda$, i.e., S is $(\frac{1}{2}, \frac{1}{2})$. If B is massive, then the little group is $SO(3)$, and S is $0 \oplus 1$ reresentation of the rotations. So f automatically vanishes unless the spin of B is 0 or 1. If B is massless, then the little group is $SE(2)$ (two dimensional Euclidean group). If D is faithful representation of $SE(2)$, then it is infinite dimensional ($SE(2)$ is noncompact), so f vanishes (S is finite dimensional). If the translations in $SE(2)$ are trivially represented (this is the case of all known massless particles), then S must be trivial for a nonvanishing f, because any other finite dimensional representation of the Lorentz group is faithful. The realization is that for a massless B the expectation value (22) can be nonzero only if the helicity of B is zero. In both cases (massless and massive) f is determined up to a constant factor because the trivial representation of the little group appears only once in $D^* \otimes S$.

Let us calculate the contribution of the one particle states B_p of mass m and spin (helicity) zero to the two point function (19). From (24) f is simply proportional to p, thus

$$\langle \text{vac}|J_a^\mu(x)|B_p^s\rangle = g_a \frac{p^\mu e^{ip\cdot x}}{(2\pi)^{\frac{3}{2}}\sqrt{p^0}},$$

where g_a is a constant (of course it does not need to be the same for different particle species). The contribution of a massless particle to the two-point function (19) (J_a^μ is supposed to be self-adjoint):

$$\int d^3\mathbf{p}\langle \text{vac}|J_a^\mu(x)|B_p\rangle\langle B_p|J_b^\nu(y)|\text{vac}\rangle = -g_a g_b^* \partial^\mu\partial^\nu\varphi(x-y), \tag{25}$$

where

$$\varphi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{e^{ip\cdot x}}{p^0}.$$

Since $\varphi(\lambda x) = \lambda^{-2}\varphi(x)$, the function φ is homogeneous, and this property is preserved by the Wick-rotation. (Later we will give an upper bound to the Euclidean two-point function.)

II. THE DIRAC OPERATOR

A. Kato inequality

Consider the Klein Gordon operator $K^A = -\sum_{\mu=0}^{d-1} D_\mu D^\mu + m^2$ for a particle of mass m and spin 0 in a given background gauge field A, where $D^\mu = \partial^\mu - igt_a A_a^\mu$. (We use the notation $A^\mu = A_a^\mu t_a$ and $F^{\mu\nu} = F_a^{\mu\nu} t_a$.) Let $\Delta_+^A(x,y;m^2)$ be the propagator of K^A. Kato's inequality asserts that Δ_+^A is bounded by the free propagator if the spacetime metric is Euclidean:

$$\|\ \Delta_+^A(x,y;m^2)\ \| \leqslant |\Delta_+^{A=0}(x,y;m^2)|,$$

where $\|\cdot\|$ is the usual matrix norm ($\|A\| = \min_{\|x\|=1}\|Ax\|$). The proof is based on the proper time path integral representation of the propagator:

$$\begin{aligned}\Delta_+^A(x,y;m^2) &= \langle x|(K^A)^{-1}|y\rangle = \frac{1}{2}\int_0^\infty dT\langle x|e^{-\frac{1}{2}TK^A}|y\rangle \\ &= \frac{1}{2}\int_0^\infty dT\int_{\xi(0)=x}^{\xi(T)=y} d\xi^\mu(t)e^{-\frac{1}{2}\int_0^T dt\frac{d\xi^\mu}{dt}\frac{d\xi_\mu}{dt}}e^{-\frac{1}{2}m^2T}\,\mathcal{T}e^{i\int_0^T A_\mu(\xi(t))\frac{d\xi^\mu(t)}{dt}dt}.\end{aligned} \tag{26}$$

$\mathcal{T}$ means time ordering. In the last equality we used that $(1/2)K^A$ can be considered as a Hamiltonian in $d+1$ dimensional spacetime, so $\exp(-(1/2)TK^A)$ is nothing but the Wick-rotated time evolution operator, and we wrote it as a path integral over all possible paths with the weight function $\exp(-S)$, where S is the classical Euclidean action. Apart from the last factor the integrand is positive (the spacetime is Euclidean), so

$$\| \Delta_+^A(x,y;m^2) \| \leqslant |\Delta_+^{A=0}(x,y;m^2)| \max_{\xi} \| \mathcal{T} e^{i\int_0^T A_\mu(\xi(t))\frac{d\xi^\mu(t)}{dt}dt} \| .$$

The proof is completed because $\mathcal{T}\exp(i\int_0^T A_\mu(\xi(t))\frac{d\xi^\mu(t)}{dt}dt)$ is unitary, so its norm is 1 for any path ξ. The above argument fails for a particle of spin $1/2$. In this case we have an extra factor in (26): $\mathcal{T}e^{(1/2)\int_0^T \sigma_{\mu\nu}F^{\mu\nu}(\xi(t))dt}$, where $\sigma_{\mu\nu} = (i/4)[\gamma_\mu, \gamma_\nu]$ are the Hermitian spin matrices (the spacetime is Euclidean). This matrix is not unitary, its norm can be bigger than 1. Vafa and Witten proved a similar inequality for the spin-$1/2$ case, but we postpone its presentation because it uses the causality of the Dirac equation, and we would rather make some comments on this issue beforhand.

B. Causality of the Dirac equation

A differential equation is hyperbolic if it has a retarded Green function (propagator) (i.e. a Green function satisfying $G(x,y) = 0$ if $x^0 < y^0$.) A hyperbolic equation on a Minkowskian spacetime is causal if its propagator vanishes for spacelike $x - y$. The free Klein-Gordon and Dirac equations are causal. From Kato's inequality this holds for the Klein-Gordon equation in the presence of a gauge field. Zwanziger gives a (nonrigorous) proof for the causality of the Diarc eqaution in the presence of an Abelian gauge (electromagnetic) field.[4] His method cannot be straightforwardly generalized to a non-Abelian background (replacing the exponentials with path ordered exponentials does not work, and the gauge invariant equation he obtains is no longer gauge invariant in the non-Abelian case, however, the latter is irrelevant in the proof of the causality). Since we have not found any argument to confirm it, we simply accept that the Dirac equation is causal even if the background is non-Abelian because we cannot do anything better.

C. Vafa-Witten inequality

Let $|\alpha\rangle$ and $|\beta\rangle$ be two states of disjoint support separated by a minimum distance R. Define the smeared Dirac propagator in the background gauge field A:

$$S_+^A(\alpha,\beta;m) := \langle\alpha|(\not{D} + m)^{-1}|\beta\rangle = \int_0^\infty d\tau e^{-m\tau}\langle\alpha|e^{-i\tau(-i\not{D})}|\beta\rangle. \qquad (27)$$

We have been thinking of $i\not{D}$ as a Euclidean Dirac operator in four dimensions, but it can be regarded as a Dirac Hamiltonian in 4+1 dimensional Minkowskian spacetime. Therefore, in (27) the factor $\exp(-i\tau(-i\not{D}))$ is the evolution operator for a real time τ in 4 + 1 dimensions. Causality implies that the state $\exp(-i\tau(-i\not{D}))|\beta\rangle$ can have an overlap with $|\alpha\rangle$ only after a time $\tau = R$. Hence we can write

$$S_+^A(\alpha,\beta;m) = \int_R^\infty d\tau e^{-m\tau}\langle\alpha|e^{-i\tau(-i\not{D})}|\beta\rangle.$$

Using Schwartz inequality for $|\alpha\rangle$ and $\exp(-i\tau(-i\not{D}))|\beta\rangle$, by unitarity of $\exp(-i\tau(-i\not{D}))$ (the four dimensional Euclidean $i\not{D}$ is Hermitian) we have

$$S_+^A(\alpha,\beta;m) \leqslant \frac{e^{-mR}}{R}\sqrt{\langle\alpha|\alpha\rangle}\sqrt{\langle\beta|\beta\rangle}. \tag{28}$$

III. VAFA-WITTEN THEOREM

Theorem. *In a gauge theory with only gauge bosons and fermions, and with $\theta = 0$, a global non-Abelian symmetry cannot be spontaneously broken if it acts nontrivially only on massive fermions.*

Proof. Let us denote the symmetry group in question with G. Consider the Euclidean two point function $\langle J_a^\mu(x)J_b^\nu(y)\rangle_A$ of the current associated with G in a given background gauge field A. The current takes the form of $J_a^\mu = \bar{q}\gamma^\mu\tau_a q$, where q is a fermion multiplet, and τ_a is a generator of the representation of G. In one multiplet the fermions have equal mass m. Imagine what the perturbative expansion (in coordinate space) for $\langle J_a^\mu(x)J_b^\nu(y)\rangle_A$ might look like. First we can have two fermion loops. Each fermion loop includes one of the vertices $\gamma^\mu\tau_a$ at x and $\gamma^\nu\tau_b$ at y. Second we can have only one fermion loop containing both. The loops have some external gauge boson legs. By the assumption we made on the gauge fields (they are G-singlet) a fermion line has the same G-flavor at both ends, hence in the contribution of a fermion loop diagram we have a trace over the G-flavors. It implies that we can exclude the first case because if the loop contains only one τ_a, then $\mathrm{Tr}\,\tau_a = 0$.[a)] But what we have in the sum of the diagrams of the second kind is nothing but two fermion propagators, one from x to y, and another from y to x. Then we can write

$$\langle J_a^\mu(x)J_b^\nu(y)\rangle_A = -\mathrm{Tr}\left[\gamma^\mu\tau_a S_+^A(x,y;m)\gamma^\nu\tau_b S_+^A(y,x;m)\right], \tag{29}$$

where $S_+^A(x,y;m)$ is the fermion propagator from x to y in the background gauge field A.

We have no upper bound for $S_+^A(x,y;m)$ but we have one for the smeared propagator $S_+^A(\alpha,\beta;m)$. Since we want to apply (28), let us smear the current by introducing smeared fermion operators:

$$q_\Delta(x) = \frac{64}{\Delta^6}\int_{||x-x'||<\Delta} d^4x' M(x,x')q(x'), \tag{30}$$

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a) Indeed, all the generators of the representation of a semi-simple compact group are traceless because taking $\tilde{\tau} = \int_G d\mu(g)D(g)\tau D(g)^{-1}$ (where μ is the invariant measure), $\mathrm{Tr}\,\tilde{\tau} = \mathrm{Tr}\,\tau \neq 0$, so $\tilde{\tau}$ is nonzero, and $D(g)\tilde{\tau}D(g)^{-1} = \tilde{\tau}$ for any $g \in G$ in contradiction to the semi-simplicity of G.

where $M(x,x')$ was introduced in order to make q_Δ gauge-covariant:

$$M(x,x') = \left\langle x \left| \left(-D_\mu D^\mu + \frac{1}{\Delta^2} \right)^{-3} \right| x' \right\rangle = \frac{1}{2} \int_0^\infty dT T^2 \left\langle x \left| e^{-T\left(-D_\mu D^\mu + \frac{1}{\Delta^2}\right)} \right| x' \right\rangle. \tag{31}$$

It is clear from the last expression in (31) that under a gauge transformation U the matrix M changes as $U(x)M(x,x')U(x')$, so the covariance of q_Δ is guaranteed. Moreover, using the same argument as in the proof of Kato's equality for the last expression in (31), we find that $\| \langle x|(-D_\mu D^\mu + 1/\Delta^2)^{-3}|x'\rangle \| \leqslant |\langle x|(-\partial_\mu \partial^\mu + 1/\Delta^2)^{-3}|x'\rangle|$. For the free Klein Gordon operator $\langle x|(K^{A=0})^{-3}|x'\rangle$ is finite for any x and x' (that is why we had to use the negative third (or more) power of the Klein Gordon operator), and it is bounded by $1/32\pi^2 m^2$, so this is true for $M(x,x')$, too. The reason why we demand the gauge invariance of q_Δ is that later we will perform the path integral over the gauge fields in order to get the two-point function of the smeared current $J^\mu_{a\Delta}$ built out of the smeared fermion field q_Δ, so $J^\mu_{a\Delta}$ must be gauge invariant because the path integral is defined only for such operators (see Faddeev Popov method). The normalization in (31) is chosen so that in the limit of small Δ the smeared fermion operator $q_\Delta(x)$ becomes the fermion operator $q(x)$ averaged over a ball of center x and radius Δ.

Define

$$|x_\Delta\rangle = \frac{64}{\Delta^6} \int_{\|x-x'\|<\Delta} M(x,x')|x'\rangle \tag{32}$$

From (32) and the upper bound for $M(x,x')$ we can see that $\| x_\Delta \|^2$ and $\| y_\Delta \|^2$ are bounded by Δ^{-4}. The least distance from $|x_\Delta\rangle$ to $|y_\Delta\rangle$ is $R = \| x - y \| - 2\Delta$. Thus (28) tells us

$$\| S^A_+(x_\Delta, y_\Delta; m) \| \leqslant e^{-m\|x-y\|} \frac{e^{2m\Delta}}{m\Delta^4}. \tag{33}$$

If we want to get the two point function of the whole quantum field theory, we have to integrate over the gauge fields A as well. In Euclidean spacetime the vacuum expectation value of a gauge invariant operator $O(q)$ built out of only fermion fields q is

$$\begin{aligned} \langle O(q)\rangle_{\text{vac}} &= \frac{1}{Z} \int dq(x) \int dA^\mu_a(x)\, O(q)\, e^{-S} \\ &= \frac{1}{Z} \int dA^\mu_a(x)\, \langle O(q)\rangle_A \det(\not{D} + m)\, e^{-\frac{1}{2g^2}\int d^4x \,\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}}, \end{aligned} \tag{34}$$

where $\det(\not{D} + m) = \int dq(x) e^{-S^A_{\text{ferm}}} = \int dq(x) \exp(-\bar{q}(\not{D} + m)q)$, which arises because $\langle O(q)\rangle_A = \int dq(x) O(q) e^{-S^A_{\text{ferm}}} / \int dq(x) e^{-S^A_{\text{ferm}}}$. The nonzero eigenvalues of $i\not{D}$ (in a given background gauge field) are paired in the following way: if $i\not{D}\psi = \lambda\psi$, then $i\not{D}\gamma_5\psi = -\lambda\gamma_5\psi$. So if λ is an eigenvalue, $-\lambda$ is also. Thus the fermion determinant is positive:

$$\det(\not{D} + m) = \prod_\lambda (m - i\lambda) = m^z \prod_\lambda (m^2 + \lambda^2) > 0,$$

where z is the number of the zero modes. In (34) $\langle \mathcal{O}(q)\rangle_A$ is integrated with respect to the measure $d\mu = dA^\mu_a(x)\ \det(\not{D}+m)\ \exp(-1/2g^2\int d^4x\ \mathrm{Tr}\ F_{\mu\nu}F^{\mu\nu})$. Since the measure $d\mu$ is positive, if we know that $|\langle \mathcal{O}(q)\rangle_A| \leqslant N$ with some constant N that is independent of A, then the vaccum expectation value of $\mathcal{O}(q)$ obeys the same inequality:

$$|\langle \mathcal{O}(q)\rangle_{\mathrm{vac}} = \left|\frac{1}{Z}\int d\mu(x)\ \langle \mathcal{O}(q)\rangle_A\right| \leqslant N. \tag{35}$$

From (29) and (33) we have a background field independent upper bound for the two-point function of the gauge invariant smeared current $J^\mu_{\Delta a} := \bar{q}_\Delta\gamma^\mu\tau_a q_\Delta$ in the presence of a given gauge field:

$$\langle J^\mu_a(x)J^\nu_b(y)\rangle_A \leqslant C^{\mu\nu}_{ab}e^{-2m\|x-y\|}\frac{e^{4m\Delta}}{m^2\Delta^8}.$$

($C^{\mu\nu}_{ab}$ is a constant, irrelevant in our argument.) By the argument that led to (35) we conclude that the same upper bound holds for the two-point function of the current (i.e. the vacuum expectation value of the product of two current operators), which we get by performing the path integral with respect to the gauge fields:

$$\langle J^\mu_a(x)J^\nu_b(y)\rangle_{\mathrm{vac}} \leqslant C^{\mu\nu}_{ab}e^{-2m\|x-y\|}\frac{e^{4m\Delta}}{m^2\Delta^8}. \tag{36}$$

Assume that the symmetry G is spontaneosly broken. It is plausible to suppose that $\bar{q}\tau_a q$ plays the role of ϕ_n in Sec.I.C. Consequently, we have a Goldstone boson which together with the vacuum gives nonzero J^μ_a matrix element, and we found in Sec.I.D that in the presence of a massless particle the two-point function acquires a term that is homogeneous function of the separation $x-y$ (see eq. (25)). We have to smear (25) in order to compare it with (36). But it is impossible that smearing a homogeneous function as in (25) would result in an exponential decay as in (36) for every Δ. □

Finally, we summarize what would happen if we relaxed some conditions in the theorem.

- If the symmetry is Abelian, then the whole argument that led to (25) (with nonzero g) breaks down because we have no candidate for ϕ_n to start the proof in Sec.I.C. ($\bar{q}q$ is neutral, it commutes with the conserved charge.)
- If we allow $\theta \neq 0$, then the complex factor $\exp((i\theta/16\pi^2)\int d^4x\ \mathrm{Tr}\ F_{\mu\nu}F^{\mu\nu})$ in the path integral over the gauge field invalidates (35).
- If the fermions are massless, then (33) blows up. This is exactly what we expect because we know (for instance, from the Atiyah-Singer index theorem) that the massless Dirac operator $i\not{D}$ has zero egienvalues for some background field, so we cannot give a background independent upper bound to the massless Dirac propagator.

- Let us include other fields, say scalars. As long as the scalars do not have Yukawa couplings to the fermions the analysis is valid, since the Euclidean action for scalars with renormalizable self-interactions and gauge couplings only is real (so (35) still holds). With Yukawa couplings the Dirac operator has the form $\not{D} + m + g\phi$. There are background ϕ fields such that $m + g\phi$ is not bounded below, and then (33) fails. This can be circumvented by pseudoscalar Yukawa couplings, when the Dirac operator is $\not{D} + m + i\gamma_5 g\phi$. The problem is that the fermion determinant is not real, and (35) does not hold. Reality and positivity can be rescued, if the fermions in the multiplet can be paired in the following way: one couples to $i\gamma_5\phi$ and the other to $-i\gamma_5\phi$. In that case the positivity of the determinant can be proved by an argument similar to one we used earlier.
- If the gauge fields transform nontrivially under the symmetry in question, then the argument that led to the conclusion that the two point function in a given background field is basically the product of two fermion propagators (see eq. (29)) fails.

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